# Noncommutative Regime of Fundamental Physics

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## Abstract

We further develop a model unifying general relativity with quantum mechanics proposed in our earlier papers (J. Math. Phys. **38**, 5840 (1998); **41**, 5168 (2000)). The model is based on a noncommutative algebra $\mathcal{A}$ defined on a groupoid $\Gamma = E \times G$ where $E$ is the total space of a fibre bundle over space-time and $G$ a Lie group acting on $E$. In this paper, the algebra $\mathcal{A}$ is defined in such a way that the model works also if $G$ is a noncompact group. Differential algebra based on derivations of this algebra is elaborated which allows us to construct a "noncommutative general relativity". The left regular representation of the algebra $\mathcal{A}$ in a Hilbert space leads to the quantum sector of our model. Its position and momentum representations are discussed in some detail. It is shown that the model has correct correspondence with the standard theories: with general relativity, by restricting the algebra $\mathcal{A}$ to a subset of its center; with quantum

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mechanics, by changing from the groupoid $\Gamma$ to its algebroid; with classical mechanics, by changing from the groupoid $\Gamma$ to its tangent groupoid. We also construct a noncommutative Fock space based on the proposed model.

# 1 Introduction

Since the seminal work by Koszul [1] it is known that the standard differential geometry (on a manifold) can be formulated in terms of a commutative associative algebra $C$, $C$-modules and connections on these modules. $C = C^\infty(M)$ is here the algebra of smooth functions on a manifold $M$, and the $C$-module is a module of smooth cross sections of a smooth vector bundle over $M$. The main idea of noncommutative geometry is to follow the above formulation of differential geometry as closely as possible with the algebra $C = C^\infty(M)$ replaced by any associative, not necessarily commutative, algebra.[2]

General relativity is a geometric theory, and — as it has been noticed by Geroch [3] — it also can be algebraically formulated according to the Koszul program. It seems quite natural to try, starting from this formulation, to create a noncommutative version of general relativity in the view of its later unification with quantum physics. The key point is the problem of metric. In general relativity the metric is a dynamical variable, and the components of the metric tensor are interpreted as gravitational potentials. The problem is that in noncommutative geometry, in general, there is no natural way of defining metric but in some cases the metric is unique and, consequently, in these cases it cannot be a dynamical variable. For instance, Madore and Mourad [4] have proved that this is true for a broad class of derivation based noncommutative differential calculi. To deal with this problem a few strategies have been elaborated.

The first of them is based on Connes' spectral calculus [5]. Let $M$ be a smooth compact $n$-dimensional manifold, and let us consider a pair $(\mathcal{A}, D)$ where $\mathcal{A} = C^\infty(M)$ and $D$ is just a "symbol" (for the time being). Let further $(\mathcal{A}_\pi, D_\pi)$ be a unitary representation of the pair $(\mathcal{A}, D)$ in a Hilbert space $\mathcal{H}_\pi$ such that the triple $(\mathcal{A}_\pi, D_\pi, \mathcal{H}_\pi)$ is a spectral triple (in the Connes sense). In such a case, there exists a unique (modulo unitary equivalence of representations $\pi$) Riemann metric $g_\pi$ on $M$ such that the geodesic distance

between any two points $p, q \in M$ is given by

$$d(p, q) = \sup_{a \in \mathcal{A}} \{|a(p) - a(q)| : \| [D_\pi, \pi(a)] \|_{\mathcal{B}(\mathcal{H}_\pi)} \leq 1\}$$

where $\mathcal{B}(\mathcal{H}_\pi)$ denotes the set of all bounded operators on $\mathcal{H}_\pi$. If the action is defined by

$$G(D) = \mathrm{tr}_\omega(D^{2-n}),$$

where $\mathrm{tr}_\omega$ is the Dixmier trace, then the unique minimum $\pi_\sigma$ is the representation of the pair $(\mathcal{A}, D)$ in the Hilbert space $\mathcal{H}_\sigma = L^2(M, S_\sigma)$ of square integrable spinors with $D_\sigma$ as the Dirac operator of the Levi-Civita connection [6].

It is often stressed by Connes that "no information is lost in trading the original Riemann manifold $M$ for the corresponding spectral triple" with the proviso that "the usual emphasis on the points $x \in M$" is now replaced by the spectrum of the Dirac operator, $\mathrm{spec}(M, D)$ with each eigenvalue repeated as required by its multiplicity.

Another approach is based on the definition of the Riemann metric as an inner product on a cotangent bundle [7]. Let us consider a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and the associated differential calculus $(\Omega_D \mathcal{A}, d)$ where $\Omega_D \mathcal{A}$ is the graded algebra of Connes' forms over the involutive algebra $\mathcal{A}$ (which is assumed to have unit) with the Connes deferential $d : \Omega_D^p \mathcal{A} \to \Omega_D^{p+1} \mathcal{A}$. In particular $\Omega_D^0 \mathcal{A} = \mathcal{A}$, and the space $\Omega_D^1 \mathcal{A}$ is the analogue of the space of cross sections of the cotangent bundle.

The spectral triple $(\mathcal{A}, \mathcal{H}, D)$ uniquely determines the canonical Hermitian structure $\Omega_D^1 \mathcal{A} \times \Omega_D^1 \mathcal{A} \to \mathcal{A}$ by

$$\langle \alpha, \beta \rangle_D := P_0(\alpha^* \beta) \in \mathcal{A}$$

for $\alpha, \beta \in \Omega_D^1 \mathcal{A}$, where $P_0$ is the orthogonal projector onto $\mathcal{A}$ as determined by the inner product on (the completion of) $\Omega_D \mathcal{A}$ defined by $(\alpha, \beta)_0 := \mathrm{tr}_\omega(\alpha^* \beta |D|^n)$ where $\mathrm{tr}_\omega$ is the Dixmier trace. The above Hermitian structure naturally extends to

$$\langle ., . \rangle_D : \Omega_D^p \mathcal{A} \times \Omega_D^p \mathcal{A} \to \mathcal{A}.$$

This Hermitian structure is weakly nondegenerate, i. e., $\langle \alpha, \beta \rangle_D = 0$, for all $\alpha \in \Omega_D^1 \mathcal{A}$ implies $\beta = 0$, and we also assume that if $(\Omega_D^1)'$ is a dual module, one has the isomorphism (of right $\mathcal{A}$-modules) $\Omega_D^1 \mathcal{A} \to (\Omega_D^1 \mathcal{A})'$ by

$\alpha \mapsto \langle \alpha, . \rangle_D$. It can be shown that such a Hermitian inner product is in fact a *Riemannian metric* on $\Omega_D^1$. One then defines the linear connection, develops the corresponding differential geometry, and constructs the Hilbert-Einstein action for "noncommutative gravity"[7].

One can also define a metric in terms of derivations of a given algebra. Noncommutative differential algebra based on derivations was developed by Dubois-Violette [8]. It was chosen by Madore to elaborate a noncommutative version of classical gravity [9]. If $\mathcal{A}$ is any associative involutive algebra with unit, one can construct over $\mathcal{A}$ a universal differential calculus $(\Omega_u \mathcal{A}, d_u)$ such that any other differential calculus over $\mathcal{A}$ can be obtained as a quotient of it. Let $(\Omega \mathcal{A}, d)$ be another differential calculus over $\mathcal{A}$. Then there exists a unique $d_u$-homomorphism $\phi : \Omega_u \mathcal{A} \to \Omega \mathcal{A}$ given by $\phi(d_u f) = df$, $f \in \mathcal{A}$, and if we know how to construct the $\mathcal{A}$-module $\Omega_1 \mathcal{A}$ and the mapping $d : \mathcal{A} \to \Omega^1 \mathcal{A}$ (satisfying the Leibniz rule), then there is a method of constructing all $\Omega^p \mathcal{A}$, for $p \geq 2$, and suitably extending the differential $d$.[10]

The idea of Madore is to define $\Omega^1 \mathcal{A}$ with the help of derivations. He assumes that derivations are internal (which implies that $\mathcal{A}$ is noncommutative). Let, for any $n \in \mathbf{N}$, $\lambda_i$ be a set of $n$ linearly independent anti-Hermitian elements of $\mathcal{A}$. Then the derivation of $\lambda_i$ is defined to be $e_i = \text{ad}\lambda_i$. We assume that if $f \in \mathcal{A}$ commutes with all $\lambda_i$ then $f$ belongs to the center of $\mathcal{A}$. The differential $d : \mathcal{A} \to \Omega_1 \mathcal{A}$ is defined by

$$df(e_i) = e_i f = [\lambda_i, f],$$

and the $\mathcal{A}$-bimodule $\Omega^1 \mathcal{A}$ is generated by all elements of the form $fdg$ (or $(df)g$). We further assume that there exist $n$ elements $\theta^i \in \Omega^1 \mathcal{A}$ such that

$$\theta^i(e_j) = \delta^i_j;$$

they are called a *frame*. If a frame exists then $\Omega^1 \mathcal{A}$ is a free (left or right) module of rank $n$. If this is the case, the construction proceeds essentially according to the general scheme.

The metric is defined by the equation

$$g(\theta^i \otimes \theta^j) = g^{ij}$$

where $g^{ij} \in \mathcal{A}$. The metric $g$ is assumed to be bilinear, and for $f \in \mathcal{A}$ one has

$$fg^{ij} = g(f\theta^i \otimes \theta^j) = g(\theta^i \otimes \theta^j f) = g^{ij} f$$

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which means that the coefficients $g^{ij}$ belong to the center $\mathcal{Z}(\mathcal{A})$ of $\mathcal{A}$. If the center $\mathcal{Z}(\mathcal{A})$ is trivial, then the coefficients $g^{ij}$ cannot be functions of coordinates, and the metric is essentially unique. Commenting on this result, Madore says that "the classical gravitational field and the noncommutative nature of space-time are two aspects of the same thing"[11]. Indeed, if we identify the metric with the gravitational field and the metric is uniquely determined by the noncommutative differential calculus, then each such differential calculus implies the unique gravitational field. This is true provided that the center of the algebra is trivial. If this is not the case, the differential calculus does not necessarily determine the metric, and the larger the center, the larger the degree of this indeterminacy. Therefore, in the limit of a commutative algebra (where the center coincides with the entire algebra) there is no obvious connection between the differential calculus and a metric.

In a series of works [12] we have proposed another approach in which a noncommutative algebra is defied on a groupoid $\Gamma = E \times G$ where $E$ is the total space of a principal fibre bundle and $G$ is a Lie group acting on $E$. The metric and the rest of differential geometry are based on derivations of this algebra. The structure of the groupoid allows for a conceptually transparent unification of general relativity and quantum mechanics. The point is that the algebra on the groupoid can be naturally made a $C^*$-algebra, and then the system can be quantized in the usual algebraic way (à la Haag and Kastler). Essentially, the $E$-part of the model is responsible for generally relativistic effects, and the $G$-part of the model for quantum effects. Being noncommutative our $C^*$-algebra defines a nonlocal space. In this way, all problems with infinities connected with sharp localization of quantum entities are a priori avoided. The model naturally explains effects due to correlations between distant events such as the EPR type of experiments in quantum mechanics and the horizon problem in cosmology. Another nice feature of the model is that it unifies concepts which seemed to be independent of each other. For instance, our $C^*$- algebra generates a von Neumann algebra which defines both a generalized dynamics and a generalized probabilistic measure (the dynamics is probabilistic, in a generalized sense, from the very beginning). When this dynamics is viewed from the quantum mechanical perspective it looks like the usual Schrödinger unitary evolution, but when it is viewed from the perspective of space-time geometry it looks like a reduction of the state vector[13].

In the present work we substantially develop this model but, to make the

paper self-consistent, we also briefly summarize the previous results. In Sec. II, we define the algebra $\mathcal{A}$ on the groupoid $\Gamma$ and propose the "unitization procedure" which ensures the existence of the correct classical limit. Differential algebra based on derivations of the algebra $\mathcal{A}$ is developed in Sec. III. This allows us to construct a noncommutative counterpart of general relativity (Sec. IV). The left regular representation of the algebra $\mathcal{A}$ in a Hilbert space leads to the quantum sector of our model (Sec. V.A). By using this representation we are able to construct "general relativity" in terms of operators on this Hilbert space (Sec. V.B). In spite of the fact that our model is nonlocal (and consequently, it has no time in the usual sense), one can define in it a generalized dynamics in terms of derivations of the algebra $\mathcal{A}$ which play the role of integral vector fields of the considered system (Sec. V.C). Then we discuss the position and momentum operators (Secs. V.D and V.E), and we comment on the position and momentum representations of our model (Sec. V.F). In section VI, we show that the model has the correct correspondence with the standard theories, i. e., with general relativity (by restricting the algebra $\mathcal{A}$ to a subset of its center), with quantum mechanics (by changing from the groupoid $\Gamma$ to its algebroid), and with classical mechanics (by changing from the groupoid $\Gamma$ to its tangent groupoid). In our view, the main drawback of the proposed model is that, although it nicely unifies general relativity with quantum mechanics, it is lacking the quantum field theoretical aspect. To at least partially improve this situation we construct, in Sec. VII, the Fock space for this model and define noncommutative counterpart of the standard operators on it.

## 2   An Algebra on a Transformation Groupoid

Let $E$ be the total space of a principal fibre bundle such that the orbits of the action of the structural group $G$ on $E$ form a differential manifold $M$, interpreted as space-time. If we want to take into account space-time singularities we must assume that $M$ is a differential (or structured) space.[14] $G$ acts (to the right) on $E$, $E \times G \to E$. We shall regard $\Gamma = E \times G$ as a groupoid. The composition of elements $\gamma_1 = (x, h)$ and $\gamma_2 = (xh, k)$ is defined to be $\gamma = \gamma_2 \circ \gamma_1 = (x, hk)$, $x \in E$, $h, k \in G$. The source map $s$ and the range map $r$ are

$$s(x, h) = (x, e), \quad r(x, h) = (xh, e)$$

where $e$ is the neutral element of $G$. We define

$$\Gamma^{(p,e)} = \{\gamma \in \Gamma : r(\gamma) = (p,e)\},$$

and dually

$$\Gamma_{(p,e)} = \{\gamma \in \Gamma : s(\gamma) = (p,e)\}.$$

In the following we shall abbreviate $\Gamma^{(p,e)}$ and $\Gamma_{(p,e)}$ to $\Gamma^p$ and $\Gamma_p$, respectively.

**Lemma 2.1** $\Gamma^p$ *and* $\Gamma_p$ *are groups isomorphic with $G$.* ∎

**Proof.** We shall show this for $\Gamma^p$. The composition in $\Gamma^p$ is

$$(ph^{-1}, h) \circ (pk^{-1}, k) = (p(hk)^{-1}, hk) \in \Gamma^p,$$

and taking the inverse

$$(ph^{-1}, h)^{-1} = (p(h^{-1})^{-1}, h^{-1}) \in \Gamma^p;$$

therefore, $\Gamma^p$ is indeed a group. Let us consider the mapping $\varphi : \Gamma \to \Gamma^p$ given by $\varphi(h) = (ph^{-1}, h)$. The mapping $\varphi$ is a bijection. Indeed, let $\varphi(h_1) = \varphi(h_2)$. Hence, $(ph_1^{-1}, h_1) = (ph_2^{-1}, h_2)$ which implies $h_1 = h_2$. The mapping $\varphi$ is also an epimorphism. Indeed, let $\gamma \in \Gamma^p \Rightarrow \gamma = (ph^{-1}, h)$. Hence, $\varphi(\gamma) = \gamma$. □

It can be readily checked that $\Gamma$ is a Lie groupoid. We recall that a *Lie groupoid* (or a *smooth groupoid*) is a groupoid $\Gamma$ such that $\Gamma$ is a manifold; $\Gamma^0$ (in our case $\Gamma^0 = E$) is a Hausdorff submanifold of $\Gamma$; each $\Gamma^p$, $p \in \Gamma^0$, is Hausdorff in the relative topology; the product and inversion maps, and the source and range maps are submersions.[15]

Now, we define the algebra $\mathcal{A}_c = C_c^\infty(\Gamma, \mathbf{C}) \cup \mathcal{A}_G$, where $C_c^\infty(\Gamma, \mathbf{C})$ is the family of compactly supported, smooth, complex valued functions on $\Gamma$, and $\mathcal{A}_G$ is the family of compactly supported, smooth, compex valued functions lifted to $\Gamma$ from the group $G$, i. e., $\mathcal{A}_G = \pi_G^*(C_c^\infty(G, \mathbf{C}))$ with $\pi_G : \Gamma \to G$ the natural projection. Let $a, b \in \mathcal{A}_c$; multiplication in $\mathcal{A}_c$ is defined to be the convolution

$$(a * b)(\gamma) = \int_{\Gamma_p} a(\gamma_1) b(\gamma_1^{-1}\gamma) d\gamma_1$$

for every $\gamma \in \Gamma_p$, $p \in E$; $d\gamma_1$ is a Haar measure.

If there are problems with the non-Hausdorff behavior in the groupoid $\Gamma$ (as it could be the case when we consider space-times with stronger types

of singularities), we can take as the algebra $C_c^\infty(\Gamma, \mathbf{C})$ the span of complex valued functions $a$ that are smooth (or continuous) with compact support on an open Hausdorff subset such that each $a$ is defined to vanish outside that open Hausdorff subset. The fact that we are considering locally compact groupoids ensures that there is "enough" of such subsets (see Ref. 15, p.31-32).

In the following, the important role will be played by the subalgebra $\mathcal{A}_{proj} := pr_M^*(C^\infty(M))$ where $pr_M = \pi_M \circ \pi_E$ with $\pi_E : \Gamma \to E$ and $\pi_M : E \to M$ being the canonical projections. As we shall see below, this subalgebra is needed to ensure to our model the correct transition to the classical case. $\mathcal{A}_{proj}$ is obviously isomorphic with the algebra of smooth functions on $M$. To incorporate it into the algebra on the groupoid $\Gamma$ we apply the following procedure.

Having any involutive algebra $\mathcal{A}$ it is always possible to construct another algebra $\mathcal{A}^+ = \mathcal{A} \times \mathbf{C}$ with the addition

$$(a, \lambda) + (b, \mu) = (a + b, \lambda + \mu)$$

and multiplication

$$(a, \lambda) \cdot (b, \mu) = (ab + \lambda b + \mu a, \lambda\mu),$$

for $a, b \in \mathcal{A}$, $\lambda, \mu \in \mathbf{C}$. Involution is defined by

$$(a, \lambda)^* = (a^*, \bar\lambda).$$

The unit in $\mathcal{A}^+$ is $\mathbf{1} = (0, 1)$, and if $\mathcal{A}$ has a norm, the norm in $\mathcal{A}^+$ can be defined by

$$\| (a, \lambda) \| = \max\{\| a \|, |\lambda|\}.$$

It can be readily shown that $\mathcal{A}$ is an ideal in $\mathcal{A}^+$. If $\mathcal{A}$ has no unit then $\mathcal{A}^+$ is isomorphic to the algebra $\tilde{\mathcal{A}}$, where $\tilde{\mathcal{A}}$ is the *minimal unitization* of $\mathcal{A}$, i. e., the minimal unital algebra containing $\mathcal{A}$.[16] We shall repeat this procedure for the algebra $\mathcal{A}_c$ over each fiber of the groupoid $\Gamma$. First, we define the algebra of complex valued functions which are constant on the fibres $\Gamma_p$ for every $p \in E$

$$\mathcal{A}_{const} = \{f \in C^\infty(\Gamma, \mathbf{C}) : f_p = \text{const}, \forall p \in E\}$$

where $f_p$ denotes $f|\Gamma_p$, and the (bilateral) action of $\mathcal{A}_{const}$ on $\mathcal{A}_c$, $\mathcal{A}_c \times \mathcal{A}_{const} \to \mathcal{A}$, is given by $(a, f) \to a \cdot f$, $(f, a) \to f \cdot a$ for $a \in \mathcal{A}_c$, $f \in \mathcal{A}_{const}$; of course, $a \cdot f = f \cdot a$.

Now, we define the algebra $\mathcal{A} = \mathcal{A}_c \times \mathcal{A}_{const}$ with the following operations

$$(a_1, f_1) + (a_2, f_2) = (a_1 + a_2, f_1 + f_2),$$

$$(a_1, f_1) * (a_2, f_2) = (a_1 * a_2 + f_1 a_2 + f_2 a_1, f_1 f_2),$$

$$(a, f)^* = (a^*, \bar{f}).$$

We shall also use the additive notation by writing $(a + f)$ instead of $(a, f)$. In this way, we obtain the involutive algebra $\mathcal{A} = \mathcal{A}_c \times \mathcal{A}_{const}$ with unit $\mathbf{1} = (0 + \mathbf{1})$ where $\mathbf{1}$ is a constant function having the value 1.

If we restrict this procedure to a single fiber $\Gamma_p$, $p \in E$, it is the minimal unitization of the algebra $\mathcal{A}_c$ restricted to this fiber.[16] If we perform this procedure for the algebra on the entire groupoid $\Gamma$, it is a non-minimal unitization of the algebra $\mathcal{A}_c$ (since we add to $\mathcal{A}_c$ not only constants, but also functions constant on fibres of $\Gamma$).

Since the algebra $\mathcal{A}$ plays the crucial role in our model we shall study some of its properties. First, we shall demonstrate that if the group $G$ is noncompact, the functions of $\mathcal{A}_{const}$ can be thought of as a limit of functions having compact supports on fibres $\Gamma_p$, $p \in E$ (if $G$ is compact, this is trivially true). To show this we first prove the following lemma.

**Lemma 2.2** *On a noncompact Lie group $G$ there is a sequence of continuous compactly supported functions $(\phi_n)_{n\in\mathbf{N}}$, $\mathrm{Im}\phi_n \in [0, 1]$ such that*

1. $\mathrm{supp}\phi_n \subset \mathrm{supp}\phi_{n+1}$,

2. $\lim_{n\to\infty} \mu(\mathrm{supp}\phi_n) = +\infty$ *where $\mu$ is the Haar measure on $G$,*

3. $\lim_{n\to\infty} \mu(\phi^{-1}(1)) = +\infty$,

4. $\lim_{n\to\infty} b_n = 0$ *where $b_n = \mu(\mathrm{supp}\phi_n \setminus \phi_\mu^{-1}(1))$,*

5. $\lim_{n\to\infty} \phi_n(g) = 1$ *for $g \in G$.* ∎

All limits are understood in the pointwise sense. Condition (4) says that the functions $\phi_n$ are equal to "almost 1" on their supports.

**Proof.** It is evident that on $\mathbf{R}$ there exists a sequence $(\lambda_n)_{n \in \mathbf{N}}$ of smooth functions $\lambda_n : \mathbf{R} \to \mathbf{R}$ satisfying conditions (1) - (5). In $\mathbf{R}^k$ we choose the sequence $(\bar{\lambda}_n)_{n \in \mathbf{N}}$ where $\bar{\lambda}_n : \mathbf{R}^k \to \mathbf{R}$ is defined by $\bar{\lambda}_n(x_1, \ldots, x_k) = \lambda_n(x_1) \cdot \lambda_n(x_2) \cdots \lambda_n(x_k)$. The sequence $(\bar{\lambda}_n)_{n \in \mathbf{N}}$ satisfies conditions (1) - (5) of the Lemma on $\mathbf{R}^k$.

Let $G \subset \mathbf{R}^k$ be a noncompact Lie group, and let us define the mapping $F : G \to \mathbf{R}^k$ by $F(g) = g$ for $g \in G$. $F$ is an isometric embedding (with respect to the natural Riemannian metric). It can be readily shown that $\phi_n = F^* \bar{\lambda}_n$ satisfies conditions (1) - (5). $\square$

**Proposition 2.1** *Every function of $\mathcal{A}_{const}$ is a (pointwise) limit of a sequence of functions with compact supports on every fiber $\Gamma_p$, $p \in E$, which satisfy conditions (1) - (5) of Lemma (2.2).* ∎

**Proof.** Let $f \in C^\infty(E)$. Then $\pi_E^* f \in \mathcal{A}_{const}$. Let further $(\phi_n)_{n \in \mathbf{N}}$ be a sequence of smooth functions on $G$ satisfying conditions (1) - (5) of Lemma (2.2). In such a case, the sequence $\psi_n = f \circ \pi_E \cdot \phi_n \circ \pi_G$, where $\pi_E : \Gamma \to E$ is the projection defined by $\pi_E(\gamma) = p$ for $\gamma \in \Gamma_p$, is the sequence of functions satisfying the conditions of the Proposition and $\lim_{n \to \infty} \psi_n(\gamma) = (\pi_E^* f)(\gamma) \in \mathcal{A}_{const}$ for $\gamma \in \Gamma$. $\square$.

We also have

**Lemma 2.3** $\mathcal{A}_{proj} \subset \mathcal{A}_{const}$. ∎

**Proof.** Let us assume that $f \in \mathcal{A}_{proj}$. We define the equivalence relation "to be in the same fiber" $\gamma_1 \sim \gamma_2 \Leftrightarrow r(\gamma_1) = r(\gamma_2)$. If $\gamma_1 \sim \gamma_2$ then $pr_M(\gamma_1) = pr_M(\gamma_2)$, and consequently $f(\gamma_1) = f(\gamma_2)$ which means that $f \in \mathcal{A}_{const}$. $\square$

The subalgebra $\mathcal{A}_{proj}$ is clearly commutative; it belongs to the center $\mathcal{Z}(\mathcal{A})$ of the algebra $\mathcal{A}$.

**Lemma 2.4** *The algebra $\mathcal{A}_{const}$ is isomorphic with the differential structure $\mathcal{A}/\Gamma$ on the space of fibres $\mathcal{G} = \bigcup_{p \in E} \Gamma_p$.* ∎

**Proof.** Let us define the mapping $\mathcal{A}_{const} \ni f \mapsto \bar{f} \in \mathcal{A}/\Gamma$ by $\bar{f}([\gamma]) = f(\gamma)$. One evidently has $\bar{f} \circ \pi_G = f$ where $\pi_G : \Gamma \to \mathcal{G}$ is the projection given by $\pi_G(\gamma) = [\gamma]$.

On the other hand, we have the inverse mapping $\mathcal{A}/G \ni \bar{f} \mapsto \bar{f} \circ \pi_G = f \in \mathcal{A}_{const}$, i.e., $f \mapsto \bar{f}$ gives us the isomorphism of the algebras $\mathcal{A}_{const}$ and $\mathcal{A}/G$. $\square$

# 3  Differential Geometry of the Groupoid

## 3.1  Metric Structure

In this subsection we consider the $\mathcal{Z}(\mathcal{A})$-submodule of derivations $\mathrm{Der}\mathcal{A}$ of the algebra $\mathcal{A} = \mathcal{A}_c \times \mathcal{A}_{const}$ of the form

$$V = \mathrm{Der}\mathcal{A}_c \oplus \mathrm{Der}\mathcal{A}_{const} \equiv V_1 \oplus V_2$$

and we define

$$(u \oplus v)(a, f) = (u(a), v(f)).$$

The pair $(\mathcal{A}, V)$ is called *differential algebra*; it will constitute the basis for the noncommutative geometry of the groupoid $\Gamma$ which we develop in the subsequent sections. At the present stage of this model development, the choice of the differential geometry $(\mathcal{A}, V)$ is the matter of convention, and it has been made mainly for the sake of simplicity. Another "natural" choice would be to consider the $\mathcal{A}$-module of all derivations of the algebra $\mathcal{A}$. This would give us much richer geometry (but also, in various places, open for non-unique generalizations); we hope to deal with this case in the future.

By the *metric in the $\mathcal{Z}(\mathcal{A})$-module $V$* we understand a $\mathcal{Z}(\mathcal{A})$-bilinear mapping $g : V \times V \to \mathcal{Z}(\mathcal{A})$. If $V$ is a free module, a signature (also a Lorentz signature) can be assigned to $g$ (it can be shown that one can also consistently consider the case when $g$ is degenerate[17]).

In our case, the structure of the $\mathcal{Z}(\mathcal{A})$-module $V$ implies the following form of the metric

$$g(u_1 + u_2, v_1 + v_2) = \overset{1}{g}(u_1, v_1) + \overset{1,2}{g}(u_1, v_2) + \overset{2,1}{g}(u_2, v_1) + \overset{2}{g}(u_2, v_2) \qquad (1)$$

where $u_1, v_1 \in V_1$ and $u_2, v_2 \in V_2$.

Let us consider a 1-form

$$\theta : \mathrm{Der}\mathcal{A} \to \mathcal{Z}(\mathcal{A}_c) \otimes \mathcal{A}_{const}$$

(we remember that $\mathcal{Z}(\mathcal{A}_{const}) = \mathcal{A}_{const}$)); we have $\theta(u_1 + u_2) = \theta(\bar{u}_1) + \theta(\bar{u}_2)$ where bar denotes the suitable lifting. With natural definitions $\theta_1(u_1 + u_2) := \theta(\bar{u}_1)$, $\theta_2(u_1 + u_2) := \theta(\bar{u}_2)$, where $u_1 \in V_1$, $u_2 \in V_2$, we have $\theta = \theta_1 + \theta_2$. Therefore, we have proved the following Lemma.

**Lemma 3.1** $A^1(\mathcal{A}_c \times \mathcal{A}_{const}) = A^1(\mathcal{A}_c) \times A^1(\mathcal{A}_{const})$ *where* $A^1$ *denotes the family of 1-forms on a given algebra.* ∎

In other words, for 1-forms the "mixed components" do not appear. Since $g(u, \cdot)$ is a 1-form, the above lemma implies that the "mixed terms" in metric (1) vanish, i. e., $\overset{1,2}{g} = \overset{2,1}{g} = 0$.

If the submodules $V_1$ and $V_2$ are free, the above metric tensor can be given in terms of bases. In our case, the $\mathcal{Z}(\mathcal{A})$-module $V_2 = \mathrm{Der}\mathcal{A}_{const}$ is always free.

The algebra $\mathcal{A}$ is a Cartesian product of the commutative algebra $\mathcal{A}_{const}$ and the noncommutative algebra $\mathcal{A}_c$. The commutative part leads to many metrics which should satisfy the part of Einstein's equation coming from $\mathcal{A}_{const}$; the noncommutative part determines the unique metric (see Ref. 4), and the part of Einstein's equation that corresponds to it should be solved with respect to derivations of the algebra $\mathcal{A}_c$ (as we shall see below).

Let $V^* = \mathrm{Hom}_{\mathcal{Z}(\mathcal{A})}(\mathrm{V}, \mathcal{A})$. It is the $\mathcal{Z}(\mathcal{A})$-module of all $\mathcal{Z}(\mathcal{A})$-valued forms on $V$. We shall assume that $V$ is reflexive, i. e., $V = V^{**}$. Let $g$ be a metric in the $\mathcal{Z}(\mathcal{A})$-module $V$. We define the mapping $V \to V^*$ by

$$\Phi_g(u)(v) = g(u, v) = \overset{1}{g}(u_1, v_1) + \overset{2}{g}(u_2, v_2).$$

We can also write $\Phi_g = (\Phi_{\underset{g}{1}}, \Phi_{\underset{g}{2}})$. We shall assume that there is the inverse mapping $\Phi_g^{-1} : V^* \to V$ defined by

$$\Phi_g^{-1}(\theta) = \Phi_{\underset{g}{1}}^{-1}\theta_1 + \Phi_{\underset{g}{2}}^{-1}\theta_2.$$

Since $g$ is assumed to be nondegenerate, the mapping $\Phi_g$ is a monomorphism. In the following, the mappings $\Phi_g$ and $\Phi_g^{-1}$ play the role analogous to that of lowering and raising indices in the usual tensorial calculus. The set $V^+ :=$ Im$\Phi_g$ is the set of *invertible forms*, i. e., the set of forms such that $\Phi_g^{-1}(V^+) = V$.

## 3.2 Connection and Curvature

Now, we develop the noncommutative differential geometry of the groupoid $\Gamma$. We begin with connection. For $u = u_1 + u_2$ and $v = v_1 + v_2$, we evidently have

$[u, v] = [u_1, v_1] + [u_2, v_2]$. We can define the preconnection $\nabla^* : V \times V \to V^*$, by using the Koszul formula, in the following way

$$(\nabla_u^* V)(x) = \frac{1}{2}[u(g(v, x)) + v(g(u, x)) - x(g(u, v)+$$

$$+g(x, [u, v]) + g(v, [x, u]) - g(u, [v, x]).$$

Since both the metric $g$ and the commutator $[u, v]$ "split" we have

$$(\nabla_u^* v)(x) = (\overset{1}{\nabla}{}^*_{u_1} v_1)(x_1) + (\overset{2}{\nabla}{}^*_{u_2} v_2)(x_2).$$

Now, we define the linear connection $\nabla : \mathcal{M} \to V$, where $\mathcal{M} = \{(u, v) \in V \times V : \nabla_u^* v \in V^+\}$, by

$$\nabla = \Phi_g^{-1} \circ \nabla^* = \Phi_{\overset{1}{g}}^{-1} \circ \overset{1}{\nabla}{}^* + \Phi_{\overset{2}{g}}^{-1} \circ \overset{2}{\nabla}{}^*.$$

Hence

$$\nabla_u v = \overset{1}{\nabla}_{u_1} v_1 + \overset{2}{\nabla}_{u_2} v_2.$$

Since $\overset{2}{g}(u, v) \in \mathcal{Z}(\mathcal{A})$, in the connection $\overset{2}{\nabla}_{u_2} v_2$ only the terms with commutators remain (in the Koszul formula).

The curvature of the linear connection $\nabla$ is an operator $R : V \times V \times V \to V$ defined by

$$R(u, v)x = \nabla_u \nabla_v x - \nabla_v \nabla_u x - \nabla_{[u,v]} x$$

and

$$R(u_1 + u_2, v_1 + v_2)(x_1 + x_2) = R(u_1, v_1)x_1 + R(u_2, v_2)x_2.$$

The domain of $R$ is $\{(u, v, x) \in V \times V \times V : (v, x) \in \mathcal{M}, (u, x) \in \mathcal{M}\}$.

We assume that $V$ is a free $\mathcal{Z}(\mathcal{A})$-module, and we choose a basis and define the trace of any linear operator $T$ in the usual way, i. e., $\mathrm{tr}T = \sum_{i=1}^k T_i^i$. Of course, for any linear operator $T = T_1 + T_2$ one has $\mathrm{tr}T = \mathrm{tr}T_1 + \mathrm{tr}T_2$.

For any pair $u, v \in V$ one defines the family of operators $R(u, v) : V \to V$ by

$$R_{u,v}(x) = R(x, u)v,$$

and the Ricci curvature $\mathbf{ric} : V \times V \to \mathcal{Z}(\mathcal{A})$ by

$$\mathbf{ric}(u, v) = \mathrm{tr}R_{uv}.$$

Finally, by putting
$$\mathbf{ric}(u,v) = g(\mathbf{R}(u),v)$$
one obtains the Ricci operator[18] $\mathbf{R} : V \to V$ which also "splits", i. e., $\mathbf{R} = \mathbf{R}_1 + \mathbf{R}_2$.

# 4 Noncommutative General Relativity

We define the Einstein operator
$$\mathbf{G} \equiv \mathbf{R} + 2\Lambda\mathbf{I} : V \to V$$
and assume that the generalized Einstein equation has the form
$$\mathbf{G} = \mathbf{G}_1 + \mathbf{G}_2 = 0. \tag{2}$$

The purely geometrical nature of the usual Einstein equation is "contaminated" by the non-geometric character of the energy-momentum tensor. Since we believe in "monistically geometric" (or "pregeometric") character of the fundamental physical level, we assume that on this level the energy-momentum tensor vanishes. We hope that the material component of the cosmic stuff could be obtained from the pregeometry via some quantum effects. However, we retain the cosmological constant $\Lambda$ since, as some recent investigations suggest, it can play an important role in fundamental physics. Moreover, a simple example of our model (in which $G$ is a finite group) shows that nonvanishing $\Lambda$ is required for the consistency of the model (see Ref. 17).

Since the metric $g$ is defined on the $\mathcal{Z}(\mathcal{A})$-module V, the generalized Einstein equation should, strictly speaking, determine both $g$ and $V$. We should notice here a subtly interplay between these two structures. The "horizontal part" of the generalized Einstein's equation ($\mathbf{G}_2 = 0$) is essentially the "lifting" of the usual Einstein's equation in space-time $M$, and all derivations $v \in V_2$ satisfy it trivially. Therefore, to solve this equation means to find the metric which satisfies it, just as it is the case in the standard general relativity. On the other hand, on the submodule $V_1$ there is essentially one metric (up to homothety) and, consequently, to solve the "vertical component" of the generalized Einstein equation ($\mathbf{G}_1 = 0$) means to find all derivations $v \in V_2$ which satisfy it.

The nonlocal character of the generalized Einstein equation should be strongly emphasized. The interaction between local and global properties of the standard general relativity were discussed from the very beginnings of this theory. As it is well known, one of the main motives for Einstein to create general relativity was the idea called by him Mach's Principle. It has many not necessarily equivalent formulations, but the underlying idea is that local properties of space-time should be entirely determined by its global properties. As long discussions have finally established, this idea is only partially implemented in the theory of general relativity[19]. Very strong "anti-Machian" property in the standard geometry is the existence of the flat (Euclidean or Minkowskian) tangent space at any point of the considered manifold. The tangent space, to the great extent, determines the local properties of this manifold independently of its global properties. Since noncommutative spaces are global objects, essentially having no local properties, they could be said "maximally Machian" spaces.

One of the original Einstein's formulation of the Mach's Principle postulated that the "metric field" should be totally determined by some global properties (such as the mass distribution in space-time, see Ref. 19, p. 67). In this sense the metric $\overset{1}{g}$ in our model is fully Machian since it is uniquely (up to homothety) determined by the noncommutative differential calculus. This is, of course, not the case as far as the metric $\overset{2}{g}$ is concerned. We believe, however, that this is due to the simplified character of our model and that the future noncommutative Einstein theory will be consequently non-local (also in the "horizontal" part of the model). All problems with the Mach's Principle reappear when space-time emerges together with its rigid local properties. This happens when the original algebra $\mathcal{A}$ is restricted to the subalgebra $\mathcal{A}_{proj}$, as it will be discussed below.

# 5 Quantum Sector of the Model

## 5.1 Left Regular Representation of the Algebra $\mathcal{A}$

We have now the formulation of (generalized) general relativity in terms of the algebra $\mathcal{A}$. Since our goal is to obtain a unification of general relativity with quantum mechanics, we must relate this formulation to some mathematical structures that are employed in quantum physics. To reach this

goal we shall use the theory of the groupoid representations in a Hilbert space. Let us then define the representation $\pi_q : \mathcal{A} \to \mathrm{End}\mathcal{H}_q$ of the algebra $\mathcal{A} = \mathcal{A}_c \times \mathcal{A}_{const}$ in the Hilbert space $\mathcal{H}_q = L^2\Gamma_q$ by

$$\pi_q(a)(\xi) = a_q * \xi \tag{3}$$

where $\xi \in \mathcal{H}_q$, and $a_q$ denotes $a \in \mathcal{A}$ restricted to the fiber $\Gamma_q$, $p \in E$, of the groupoid $G$; or, more precisely,

$$\pi_q(a + f)(\xi) = \overset{1}{\pi}_q(a)(\xi) + \overset{2}{\pi}_q(f)(\xi) \tag{4}$$

where

$$\overset{1}{\pi}_q(a)(\xi) = \int_{\Gamma_q} a(\gamma_1)\xi(\gamma_1^{-1}\gamma)d\gamma_1,$$

and

$$\overset{2}{\pi}_q(f)(\xi) = f_q \cdot \xi.$$

We also define the "integrated" representation of $\mathcal{A}$

$$\pi = \bigoplus_{q \in E} \overset{1}{\pi}_q + \bigoplus_{q \in E} \overset{2}{\pi}_q \tag{5}$$

in the Hilbert space $\mathcal{H} = \bigoplus_{q \in E} \mathcal{H}_q$. Here, and above, we use the additive convention (summation is understood as a pair). The representation $\pi$ is what in the theory of groupoid representations is called the *left regular representation* of the algebra $\mathcal{A}$ in the Hilbert space $\mathcal{H}$ (see Ref. 15, Chapter 3.1).

Our aim is now to introduce a norm in the algebra $\mathcal{A}$. To do so we restrict the algebra $\mathcal{A}_{const}$ to the subalgebra $A_{const}^b$ of *bounded* functions constant on the fibres of the groupoid $\Gamma$. As we shall see later (Sec. VI.A), the bounded functions of $\mathcal{A}_{const}$ are enough to ensure the correct transition from our model to the ordinary space-time geometry. We define the norm in the algebra $\mathcal{A}$

$$\| (a, f) \| = \max\{\| a \|, \| f \|\}$$

where $\| a \| = \sup_{q \in E} \|\overset{1}{\pi}_q(a) \|$, and $\| f \| = \sup_{q \in E} \|\overset{2}{\pi}_q(f) \|$, $a \in \mathcal{A}_c$, $f \in \mathcal{A}_{const}^b$. It is also the norm when $\pi$ is restricted to $\pi_q$, i. e., to a single fibre of $\Gamma$ over $q \in E$. The algebra $\mathcal{A}$ completed with respect to this norm is a $C^*$-algebra. In the following, we shall always assume that this is the case.

Since now $\mathcal{A}$ is a $C^*$-algebra we can use it to quantize the system with the help ot the algebraic method (as it is done in Ref. 12). If $a$ is a Hermitian element of $\mathcal{A}$ and $\varphi$ a state on the algebra $\mathcal{A}$, then $\varphi(a)$ is the expectation value of the observable $a$ if the system is in the state $\varphi$. However, in the following we shall develop the quantum sector of our model in terms of operators on a Hilbert space (by using the above left regular representation of $\mathcal{A}$) rather than directly in terms of the $C^*$-algebra.

## 5.2 General Relativity in Terms of Operators on a Hilbert Space

Let us consider any representation of the algebra $\mathcal{A}$ in the Hilbert space $\bigoplus_{p \in E} L^2(\Gamma^p)$, and let us assume that $\pi$ is a monomorphism. We define $\hat{\mathcal{A}} := \pi(\mathcal{A})$, and also if $v \in \mathrm{Der}\mathcal{A}$, $\hat{v}(\pi(a)) := \pi(v(a))$ for $a \in \mathcal{A}$.

Let $A$ be a tensor of the type $(n, 0)$, $A : \mathrm{Der}\mathcal{A} \times \cdots \times \mathrm{Der}\mathcal{A} \to \mathcal{A}$. We evidently have $\hat{A}(\hat{v}_1, \ldots, \hat{v}_n) = \pi(A(v_1, \ldots, v_n))$, and similarly for $A : \mathrm{Der}\mathcal{A} \times, \cdots, \mathrm{Der}\mathcal{A} \to \mathcal{Z}(\mathcal{A})$, $\hat{A} : \mathrm{Der}\hat{\mathcal{A}} \times, \cdots, \times \mathrm{Der}\mathcal{A} \to \mathcal{Z}(\hat{\mathcal{A}})$, we also obtain $\mathcal{Z}(\hat{\mathcal{A}}) = \widehat{\mathcal{Z}(\mathcal{A})}$. And analogously, for tensors of other types. In particular, the above refers to derivations. We have $\hat{v}(\hat{a}) = \widehat{v(a)}$ for $a \in \mathcal{A}$, and since in the space of operators all derivations are internal, $\mathrm{Der}\hat{\mathcal{A}} = \{\mathrm{ad}\hat{a} : a \in \mathcal{A}\}$. Hence $\hat{v}(\hat{a}) = (\mathrm{ad}\hat{b})(\hat{a})$, or equivalently

$$\pi(v(a)) = [\pi(b), \pi(a)]. \tag{6}$$

In this way, we have shown that each tensor on derivations of the algebra $\mathcal{A}$ uniquely determines a tensor on operators (in the image of the representation $\pi$). Notice that this is valid only if $\pi$ is a monomorphism; therefore, in the case when $\pi$ is the representation of the algebra $\mathcal{A}$ considered in the preceding subsection, this is valid for the "integrated" representation $\pi = \bigoplus_{p \in E} \pi_p$, and is not valid for representations $\pi_p$, $p \in E$.

In the preceding section, we have defined the generalized Einstein equation in terms of the algebra $\mathcal{A}$; now, we are able to "transfer" this equation, with the help of the left regular representation, to the Hilbert space $\mathcal{H} = \bigoplus_q L^2(\Gamma^q)$, i. e., we are able to express the generalized Einstein equation in terms of operators on the Hilbert space $\mathcal{H}$. For instance, the (Lorentz) metric $g(u, v)$, $u, v \in \mathrm{Der}\mathcal{A}$ can be expressed in the following way

$$\hat{g}(\hat{u}, \hat{v}) = \hat{g}(\hat{u}_1 + \hat{u}_2, \hat{v}_1 + \hat{v}_2) = \overset{1}{\hat{g}}(\mathrm{ad}\hat{a}_1, \mathrm{ad}\hat{b}_1) + \overset{2}{\hat{g}}(\hat{u}_2, \hat{v}_2),$$

and similarly for other tensors appearing in the generalized Einstein equation. Finally, this equation will have the form

$$\widehat{\mathbf{G}} = 0. \tag{7}$$

It is worthwhile to notice that this formulation of the "generalized general relativity" has a strong nonlocal flavor: it can be only done on the "integrated space" $\bigoplus_q L^2(\Gamma^q)$, but not on a single "fibre" $L^2(\Gamma^q)$.

## 5.3   Generalized Quantum Dynamics

Derivations of the algebra $\hat{\mathcal{A}} = \pi(\mathcal{A})$ have the form of equation (6). If we restrict this equation to a single fibre of the groupoid $\Gamma$ over $q \in E$, we obtain

$$\pi_q(v(a)) = [\hat{b}, \pi_q(a)]. \tag{8}$$

We can see that equation (8) (resp. (6)) bears strong resemblance to the known Schrödinger equation in the Heisenberg picture of quantum mechanics, describing evolution of observables when states are constant. Indeed, equation (8) (or (6)) can also be regarded as describing the evolution of the operator $\pi_q(a)$, $a \in \mathcal{A}$ with $\hat{b}$ playing the role of a "Hamiltonian" (which in this case depends on $v$). Although in our model there is no concept of time (in the usual sense), derivations can be thought of as counterparts of vector fields, and equation (8) (or (6)) as modelling the dynamics in terms of the "integral vector field" $v \in \mathrm{Der}\mathcal{A}$. Basing on this heuristic argument we postulate that the dynamics of a quantum gravitational system is described by the following equation

$$i\hbar\pi_q(v(a)) = [F_v, \pi_q(a)] \tag{9}$$

for every $q \in E$. Here, for the sake of generality, we assume that $F_v$ is a one-parameter family of operators $F_v \in \mathrm{End}\mathcal{H}$, $\mathcal{H} = L^2(\Gamma_q)$, such that

$$F_{\lambda_1 v_1 + \lambda_2 v_2} = \lambda_1 F_v + \lambda_2 F_2$$

with $\lambda_1, \lambda_2 \in \mathbf{C}$. To connect this dynamics with Einstein equation (2) we additionally postulate that the derivations $v, v_1$ and $v_2$ be solutions of the "vertical part" of Einstein's equation, i. e., $v, v_1, v_2 \in \ker\mathbf{G}_1$. We also assume that $[F_v, \pi_q(a)]$ is a bounded operator. The coefficient $i\hbar$ has been added to

guarantee the correspondence with the standard quantum mechanics. Let us notice that in fact we have a **C**-linear mapping $\Phi : \ker\mathbf{G} \to \mathrm{End}\mathcal{H}$ satisfying

$$i\hbar\pi_q(v(a)) = [\Phi(v), \pi_q(a)]. \tag{10}$$

It should be emphasized that the above described "noncommutative dynamics" depends on the "form" $\Phi$. This remains in consonance with the result obtained in our previous work[20] in which the dynamics for our model has been introduced in terms of von Neumann algebras. These algebras turn out to be natural "dynamical objects" in noncommutative geometry (see Ref. 2, p.44). To be more precise, if the operator $F_v$ in Eq. (9) is positively defined and bounded then, on the strength of the Tomita-Takesaki theorem[21], there exists a one-parameter group $(\alpha_t^\phi)_{t\in\mathbf{R}}$ of automorphisms of the von Neumann algebra $\pi(\mathcal{A})''$ where $\pi = \bigoplus_{\pi\in E}\pi_q$ (depending on a form $\phi$ on this algebra) in terms of which dynamics can be defined (see Eq. (6) in Ref. 20).

The fact that $v \in \ker\mathbf{G}$ makes of eqs. (2) and (10) a "noncommutative dynamical system". To solve this system means to find the set

$$\mathcal{E}_\mathbf{G} = \{a \in \mathcal{A} : i\hbar\pi_q(v(a)) = [\Phi(v), \pi_q(a)], \forall v \in \ker\mathbf{G}\}.$$

It can be easily verified that it is a subalgebra of $\mathcal{A}$.

Let $\bar{\mathcal{E}}_\mathbf{G}$ be the smallest closed involutive subalgebra of the algebra $\mathcal{A}$ containing $\mathcal{E}_\mathbf{G}$. $\bar{\mathcal{E}}_\mathbf{G}$ is said to be *generated by* $\mathcal{E}_\mathbf{G}$. Since $\mathcal{A}$ is assumed to be a $C^*$-algebra and every closed involutive subalgebra of a $C^*$-algebra is a $C^*$-algebra[22], $\bar{\mathcal{E}}_\mathbf{G}$ is also a $C^*$-algebra; it will be called *Einstein $C^*$-algebra* or simply *Einstein algebra*, and the pair $(\bar{\mathcal{E}}_\mathbf{G}, \ker\mathbf{G})$ – *Einstein differential algebra*. We can also define another subalgebra of $\mathcal{A}$

$$\mathcal{E}_v := \{a \in \mathcal{A} : i\hbar\pi_q(v(a)) = [\Phi(v), \pi_q(a)]\},$$

where $v \in \ker\mathbf{G}$. It can be easily seen that $\mathcal{E}_\mathbf{G} = \bigcap_{v\in\ker\mathbf{G}} \mathcal{E}_v$. Of course, it can happen that $\mathcal{E}_v = \{0\}$ which would imply that also $\mathcal{E}_\mathbf{G} = 0$.

Dynamical equation (9) can be written in the form

$$\pi_q(v_1(a)) + (v_2(f)) = [F_{v_1+v_2}, \pi_q(a+f)]$$

where $a \in \mathcal{A}_c$, $f \in \mathcal{A}_{const}$, $v_1 \in V_1$, $v_2 \in V_2$. This is equivalent to

$$\overset{1}{\pi}_q (v_1(a)) + \overset{2}{\pi}_q (v_2(f)) = [F_{v_1}, \overset{1}{\pi}_q (a)] + [F_{v_2}, \overset{2}{\pi}_q (f)],$$

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and the last commutator vanishes since $f$ is constant on the considered fiber. Finally, we have

$$\pi_q(v_1(a)) = [F_{v_1}, \pi_q(a)]$$

and

$$\pi_q(v_2(f)) = 0.$$

The last equality implies that $(v_2(f))_q = 0$. Analogous results are valid for the "integrated representation" $\pi$.

## 5.4   Position and Momentum Operators

In this subsection we consider position and momentum operators in our "generalized quantum mechanics". Let $M$ be any relativistic space-time (4-dimensional smooth manifold). It is evident that the projection $pr : \Gamma \to M$, $pr = \pi_M \circ \pi_E$, which is clearly connected with localization in $M$, is not a numerical function (it has no values in $\mathbf{R}$ or $\mathbf{C}$), and consequently it does not belong to the algebra $\mathcal{A}$). However, if we choose a local coordinate map $x = (x^\mu)$, $\mu = 0, 1, 2, 3$, in $M$ then the projection $pr$ determines four observables in the domain $\mathcal{D}_x$ of $x$

$$pr_\mu = x^\mu \circ pr$$

We thus have the system of four position observables $pr = (pr_0, pr_1, pr_2, pr_3)$. Of course, $pr_\mu \in \mathcal{A}_{proj}(pr^{-1}(\mathcal{D}_x))$ and it is Hermitian.

Let us notice that the projection $pr : \Gamma \to M$ contains, in a sense, the information about all possible local observables $pr_\mu$. This can be regarded as a "noncommutative formulation" of the fact that there is no absolute position but only the position with respect to a local coordinate system.

Let us apply representation (4) to the position observable, e. g, to $pr_1$. Of course, $pr_1 \in \mathcal{A}_{const}(pr^{-1}(\mathcal{D}_x))$, and we have

$$(\pi_q(pr_1))(\xi) = \overset{1}{\pi}_q (0)(\xi) + \overset{2}{\pi}_q (pr_1)(\xi) = (pr_1)_q \cdot \xi = x \cdot \xi,$$

$x \in M$, in the local map. We see that the position observable in the quantum sector of our model has the same form as in the ordinary quantum mechanics. This indicates that we are working in the position representation of our model.

By analogy with the ordinary quantum mechanics a derivation of the algebra $\mathcal{A}$ should in our model play the role of the momentum operator. We shall see that this is indeed the case. However, first let us prove the following Lemma.

**Lemma 5.1** *Let $\pi : \mathcal{A} \to \mathrm{End}\mathcal{H}$ be a (nondegenerate) representation of the algebra $\mathcal{A}$ in the Hilbert space $\mathcal{H}$. Any internal derivation $\mathrm{ad}a, a \in \mathcal{A}$, of $\mathcal{A}$ has an operator representation of the form $\mathrm{ad}\pi(a)$, which we shall also denote by $\pi(\mathrm{ad}a)$. The operator $\pi(\mathrm{ad}a)$ is an element of the algebra $\pi(\mathcal{A})$ if and only if $a \in \ker\pi$.*

**Proof.** It is trivial to check that $\mathrm{ad}\pi(a)$ is a derivation of the algebra $\pi(\mathcal{H})$ (and consequently of the algebra $\mathrm{End}\mathcal{H}$). For any $\omega \in \mathrm{End}\mathcal{H}$, $\mathrm{ad}\pi(a)(\omega) = \pi(a) \circ \omega - \omega \circ \pi(a)$. If $a \in \ker\pi$, then $\pi(a) = 0$. In such a case, $\mathrm{ad}\pi(a) = \pi(a) = 0$. On the other hand, if $\pi(\mathrm{ad}a) \in \pi(\mathcal{A})$ then $\pi(\mathrm{ad}a) = \pi(b)$, or $\mathrm{ad}\pi(a) = \pi(b)$ which implies that $\pi(a) = 0$. $\square$

From this lemma it follows that the derivation, which is to be interpreted as the momentum operator, must be an element of $\mathrm{End}\mathcal{H} \setminus \pi(\mathcal{A})$; in other words, it must be an external derivation. It also must be a "lifting" of a local basis in space-time given by a local coordinate system. Let $(pr_0, pr_1, pr_2, pr_3)$ be the position observables with respect to a local coordinate system $x = (x^0, x^1, x^2, x^3)$ in space-time $M$. The lifts $\bar{\partial}_0, \bar{\partial}_1, \bar{\partial}_2, \bar{\partial}_3$ (to $\Gamma$) of the basis fields $\partial_0, \partial_1, \partial_2, \partial_3$ in $M$, corresponding to the coordinate system $x$, satisfy the condition

$$\bar{\partial}_\mu(pr_\nu) = \delta_{\mu\nu},$$

$\mu, \nu = 0, 1, 2, 3$. Let now $\hat{\partial}_0, \hat{\partial}_1, \hat{\partial}_2, \hat{\partial}_3$ be derivations in the space of operators $\mathrm{End}\mathcal{H}$, where $\mathcal{H} = L^2(\Gamma_{\mathcal{D}_x})$ and $\mathcal{D}_x$ is the domain of the coordinate map $(x_0, x_1, x_2, x_3)$. We have

**Lemma 5.2** *The following commutation relations are valid*

$$[\hat{\partial}_\mu, \pi(pr_\nu)] = \delta_{\mu\nu}\mathbf{1}.$$

**Proof.** Direct computation by assuming that $\xi \in L^2(G_{\mathcal{D}_x})$ is a function on $pr^{-1}(\mathcal{D}_x)$. $\square$

In the following Section we shall present the above results in an elegant mathematical form.
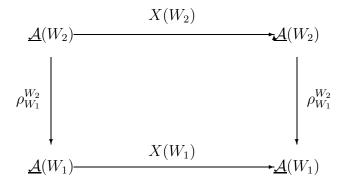
## 5.5 A Sheaf Structure on the Groupoid

On the Cartesian product $\Gamma = E \times G$ there exists the natural product topology; however, we shall consider a weaker topology in which the open sets are of the form $\pi_E^{-1}(U)$ where $U$ is open in the manifold topology $\tau_E$ on $E$. Every such open set is also open in the topology $\tau_E \times \tau_G$. Indeed, every such set is given by $\pi_E^{-1}(U) = U \times G$.

Let $\underline{\mathcal{A}}$ be a functor which with an open set $U \times G$ associates the involutive noncommutative algebra $\mathcal{A}(U \times G)$ of smooth compactly supported complex valued functions with the ordinary addition and convolution multiplication. As it can be easily seen, $\underline{\mathcal{A}}$ is a sheaf of noncommutative algebras on the topological space $(\Gamma, \pi_E^{-1}(\tau_E))$.

The projection $pr \colon \Gamma \to M$ can be *locally* interpreted as a set of (local) cross sections of the sheaf $\underline{\mathcal{A}}$ (i.e. as a set of position observables). Indeed, for the domain $\mathcal{D}_x$ of any coordinate map $x = (x^0, x^1, x^2, x^3)$, the composition $x \circ pr = (x^0 \circ pr,\ x^1 \circ pr,\ x^2 \circ pr,\ x^3 \circ pr)$ is a set of such local cross sections of $\underline{\mathcal{A}}$ on the open set $\pi_E^{-1}(\mathcal{D}_x \times G)$. The global mapping $pr : \Gamma \to M$ is not a cross section of $\underline{\mathcal{A}}$.

Let us notice that to a measurement result which is not a number but a set of numbers (a vector, a spinor... ) there does not correspond a single observable but rather a set of observables, i. e., a set of (local) cross sections of the sheaf $\underline{\mathcal{A}}$.

Now, we define the *derivation morphism* of the sheaf $\underline{\mathcal{A}}$ over an open set $U \in \pi_E^{-1}(\tau_E)$ as a family of mappings $X = (X_W)_{W \subset U}$ such that $X_W : \underline{\mathcal{A}}(W) \to \underline{\mathcal{A}}(W)$ is a derivation of the algebra $\underline{\mathcal{A}}(W)$, and for any $W_1, W_2$ open and $W_1 \subset W_2 \subset U$, the following diagram commutes

$$\begin{array}{ccc}
\underline{\mathcal{A}}(W_2) & \xrightarrow{\quad X(W_2) \quad} & \underline{\mathcal{A}}(W_2) \\
\downarrow{\scriptstyle \rho_{W_1}^{W_2}} & & \downarrow{\scriptstyle \rho_{W_1}^{W_2}} \\
\underline{\mathcal{A}}(W_1) & \xrightarrow{\quad X(W_1) \quad} & \underline{\mathcal{A}}(W_1)
\end{array}$$

where $\rho_{W_2}^{W_1}$ is the known restriction homomorphism. The family of all derivation morphisms indexed by open sets is a sheaf of $\mathcal{Z}(\mathcal{A})$-modules where $\mathcal{Z}(\mathcal{A})$ denotes the sheaf of centers of the algebras $\underline{\mathcal{A}}(U), U \in \pi_E^{-1}(\tau_e)$.

Components of the momentum observable $\bar{\partial}_\mu$ are cross sections of the sheaf of $\mathcal{Z}(\mathcal{A})$-modules of derivations of the sheaf $\underline{\mathcal{A}}$ over domains of coordinate maps, and the representation $\pi_U : \underline{\mathcal{A}}(U) \to \pi_U(\underline{\mathcal{A}}(U))$, where $U \in \pi_E^{-1}(\tau_E)$, transfers the sheaf structure from the groupoid $\Gamma$ to the family of operator algebras over the topological space $(\Gamma, \pi_E^{-1}(\tau_E))$.

## 5.6  Momentum Representation of the Model

As it was noticed above, so far we were working in the position representation of our model; in the present Subsection we shall briefly indicate how its momentum representation can be constructed. To this end, we must turn, by close analogy with the standard case, to the harmonic analysis on a groupoid.

The usual Fourier transform (on the real line) changes translations into multiplications by a function and, consequently, it enables one to perform the spectral decomposition of any operator which commutes with translations. The generalized Fourier transform plays the same role with respect to groups. Let $G$ be a topological group, and $\hat{G}$ the set of equivalence classes of irreducible unitary representations of $G$. For every $\lambda \in \hat{G}$ let $T_\lambda$ denotes a representation of $G$ in a Hilbert space $\mathcal{H}_\lambda$ which belongs to this equivalence class. Let us also assume that $G$ is a locally compact group, and let us consider $f \in L^1(G, dg)$. The operator valued function $\tilde{f} : \hat{G} \to \mathrm{End}\mathcal{H}_\lambda$ defined by

$$\tilde{f}(\lambda) = \int_G f(g) T_\lambda(g) dg$$

is said to be the *Fourier transform* of $f$ at $\lambda \in \hat{G}$.

If $G$ is a compact noncommutative group, the set $\hat{G}$ is discrete[23]. Since all irreducible representations of $G$ are finite dimensional, we can assume

that $\tilde{\lambda} \in \mathrm{Mat}_{n(\lambda)}(\mathbf{C})$ where $n(\lambda)$ is the dimension of $T_\lambda$, $\lambda \in \hat{G}$.

Let now $L^2(\hat{G})$ be the space of all matrix valued functions on $\hat{G}$ such that:
(i) $\phi(\lambda) \in \mathrm{Mat}_{n(\lambda)}(\mathbf{C})$ for every $\lambda \in \hat{G}$, (ii) $\sum_{\lambda \in \hat{\Gamma}} n(\lambda)\mathrm{tr}(\phi(\lambda)^*, \phi(\lambda)) < \infty$.
$L^2(\hat{G})$ is a Hilbert space with the scalar product

$$(\phi_1, \phi_2) = \sum_{\lambda \in \hat{\Gamma}} n(\lambda)\mathrm{tr}(\phi_1(\lambda)\phi_2(\lambda)^*).$$

It can be shown that the Fourier transform of $f \in L^1(G, dg)$ prolongs to the isometric mapping of the space $L^2(G, dg)$ onto the space $L^2(\hat{G})$ (Ref. 23, p.194).

Basing on these results we define the *dual groupoid* $\hat{\Gamma} = E \times \hat{G}$ of the groupoid $\Gamma = E \times G$. Accordingly, we have the "dual algebra" $\hat{\mathcal{A}} = \{\hat{\Gamma} \to \mathrm{End}\mathcal{H}_{\mathcal{A}}\}$ to the algebra $\mathcal{A} = \{G \to \mathbf{C}\}$. Let $a \in \mathcal{A}$, $a = (a_p)_{p \in E}$. On the strength of the Fourier transform properties to every $a_p : \Gamma_p \to \mathbf{C}$ there corresponds the function

$$\hat{a}_p : \hat{\Gamma}_p \to \bigsqcup_{\lambda \in \hat{\Gamma}_p} \mathrm{Mat}_{n(\lambda)}(\mathbf{C}),$$

where $\bigsqcup$ denotes disjoint sum, and $\hat{a} = (\hat{a}_p)_{p \in E}$. Since we have the representation $\pi_p$ of the algebra $\mathcal{A}$ in the Hilbert space $L^2(\Gamma_p)$, $p \in E$, $\pi_p \colon \mathcal{A} \to \mathrm{End}L^2(\Gamma_p)$, we obtain the Fourier transform

$$L^2(\Gamma_p) \xrightarrow{\mathcal{F}} L^2(\hat{\Gamma}_p)$$

where $L^2(\hat{\Gamma}_p)$ is the Hilbert space with the scalar product

$$(\phi_1, \phi_2) = \sum_{\lambda \in \hat{\Gamma}} n(\lambda)\mathrm{tr}(\phi_1(\lambda), \phi_2(\lambda)^*).$$

The algebra $\hat{\mathcal{A}}$ can be regarded as leading to the momentum representation of our model. In this representation a projection corresponds to the momentum operator and a derivation to the position operator. If $G$ is a compact group one can use the Peter-Weyl theorem[24] to develop the theory of dual groupoids.

# 6  Coresppondence with Standard Theories

Since our model is a unification of general relativity with quantum mechanics, we should check whether it leads to these theories as its suitable "limiting cases". We do this in the present section. Additionally, we discuss the transition from the quantum sector of our model to classical mechanics. It turns out that all these transitions beautifully fit into the structure of the model. To see the importance of such discussions we send the reader to an interesting paper by Joy Christian [25].

## 6.1  Correspondence: The Model – General Relativity

The transition from our model to the standard theory of general relativity is done essentially by restricting the algebra $\mathcal{A} = \mathcal{A}_c \times \mathcal{A}_{const}$ to the subalgebra $\mathcal{A}_{proj} \subset \mathcal{A}_{const}$. If we assume (as in Section V.A) that $\mathcal{A}_{const}$ consists only of bounded functions then $\mathcal{A}_{proj} = \pi_M^*(C_b(M))$ where $C_b(M) \subset C^\infty(M)$ denotes all smooth, bounded, complex valued functions on $M$. It is obvious that from $C^\infty(M)$ one can reconstruct the full geometry of space-time $M$. In this way, one recovers the standard theory of general relativity (see Ref. 3). We shall show that this can also be achieved if we use $C_b(M)$ instead of $C^\infty(M)$.

Let $M$ be a non empty set and $C$ a family of functions on $M$. We denote by sc$C$ the set of functions on $M$ of the form $\omega \circ (\alpha_1, \ldots, \alpha_n) \in C$ where $\omega$ is a smooth function on $\mathbf{R}^n$ and $\alpha_1, \ldots, \alpha_n \in C, n = 1, 2, \ldots$ A family $C$ is said to be *closed with respect to the smooth functions* on $\mathbf{R}^n$ if $C = \text{sc}C$.

Let $\tau_C$ be the weakest topology on $M$ in which all functions belonging to $C$ are continuous, and let $A$ be a subset of $M$. By $C_M$ we denote the set of functions $g : A \to \mathbf{R}$ such that for each point $p \in A$ there exists an open neighborhood $U$ of $p$ and a function $f \in C$ such that $f|U\cap A = g|U\cap A$. A set $C$ is said to be *closed with respect to localization* if $C_M = C$. If $C = (\text{sc}C)_M$ then the pair $(M, C)$ is called a *differential space* and $C$ a *differential structure* on $M$.

Let $(M, C)$ be a differential space. Its differential structure is said to be *finitely generated* by a set $C_0 = \{\alpha_1, \ldots, \alpha_n)$ if $(\text{sc}C_0)_M = C$.

It can be easily seen that the differential structure $C^\infty(\mathbf{R})$ of the differential space $(\mathbf{R}, C^\infty(\mathbf{R}))$ (which is, of course, a differential manifold) is generated by the identity on $\mathbf{R}$, $id_{\mathbf{R}}$, which is an unbounded function, but it can also be generated by a bounded function, for instance by $x \mapsto \text{arctg}x, x \in \mathbf{R}$.

Indeed, each of these two functions can be expressed by the other one by the composition with a function of the class $C^\infty$. The same is true for the differential space $(\mathbf{R}^n, C^\infty(\mathbf{R}^n))$. Since any smooth manifold is locally a Euclidean space, its differential structure is generated by coordinates of a local map. Their composition with the function arctg$x$ gives us the set of bounded functions that generates this differential structure. Therefore, without any loss of generality, we can assume that $\mathcal{A}_{proj}$ consists of bounded functions.

It seems rather unusual that the transition from our model to the ordinary space-time geometry has the character of a restriction of a certain algebra to its subalgebra. In physics, in analogous situations, we are usually confronted with a "smooth transition to a limit". We might, however, regard the restriction of our algebra as a "phenomenological description" of a more subtle process. Indeed, in the case when $G$ is a noncompact group, on the strength of Proposition II.1, every function $f \in \mathcal{A}_{proj}$ can be regarded as a (pointwise) limit of a sequence of functions belonging to the algebra $\mathcal{A}_c$. In this sense, also in our model, the emergence of space-time can be thought of as a "limiting process".

## 6.2 Correspondence: The Model – Quantum Mechanics

The quantum sector of our model, in spite of many similarities, in some respects differs from the standard quantum mechanics. First of all, it is strongly coupled to gravity. Formally, this has been achieved by the fact that both gravity and quantum effects are modelled by the same algebra on the groupoid $\Gamma$ (kinematic aspect), and the fact that derivations determining the dynamics of quantum operators (Eq. (9)) are postulated to be solutions of Einstein equation (2) (dynamical aspect). As the consequence of these properties, the phase space of our "generalized quantum mechanics" is the Hilbert space $\mathcal{H}_q = L^2(\Gamma_q)$ (or $\mathcal{H} = \bigoplus_{q \in E} L^2(\Gamma_q)$), and since $\mathcal{H}_q$, for all $q \in E$, is isomorphic with $L^2(G)$, in fact, we have a "quantum mechanics on the group $G$" rather than on $\mathbf{R}^n$, as it is the case for the position representation of the ordinary quantum mechanics. Our model is strongly global, therefore, we could expect that a "local version" of the model would lead to the standard quantum mechanics. This is indeed the case. The local version of the groupoid is the algebroid. We shall show that the "algebroid version"

of our model reproduces the usual quantum mechanics.

The *Lie algebroid* of a Lie groupoid $\Gamma$ is the vector bundle

$$A(\Gamma) = \bigcup_{\gamma \in \Gamma^0} T_\gamma(\Gamma^p),$$

where $\gamma = (p, e)$, with the bundle projection $\phi : A(\Gamma) \to \Gamma^0$ such that $\phi(T_\gamma(\Gamma^p)) = \gamma$. The fiber of this vector bundle over $\gamma = (p, e) \in \Gamma^0$ is, in fact, the tangent space to $p$ along the space $\Gamma^p$. The smooth structure of $A(\Gamma)$ is defined in the natural way (see Ref. 15, p. 58). It can be easily seen that when $\Gamma$ is a Lie group (which is trivially a groupoid) then $\Gamma^0 = \{e\}$, and the Lie algebroid of $\Gamma$ becomes the Lie algebra of $\Gamma$. Let us apply this to our case.

Let $U_q$ be a (starshaped) neighbourhood of the unit element $e$ of the group $G$. The exponential map

$$\psi_q \equiv \exp : T_e G \to U_q$$

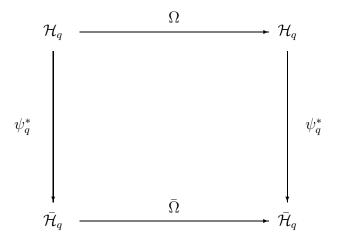induces the isomorphism of the Hilbert spaces

$$\psi_q^* : L^2(U_q) \to L^2(T_e G).$$

Indeed, the equality $\langle \xi, \zeta \rangle = \langle \xi \circ \psi_q, \zeta \circ \psi_q \rangle$, $\xi, \zeta \in L^2(U_q)$, follows from the construction of the integral on a manifold in the local map $\psi_q$.

Let us consider the mappings

$$\mathcal{A} \xrightarrow{\pi_q} \mathrm{End}(\mathcal{H}_q) \xrightarrow{J} \mathrm{End}(\bar{\mathcal{H}}_q)$$

where $\bar{\mathcal{H}}_q = L^2(T_e \Gamma^q)$, and $J$ is defined in the following way. First, let us introduce the commutative diagram

$$
\begin{array}{ccc}
\mathcal{H}_q & \xrightarrow{\quad\Omega\quad} & \mathcal{H}_q \\
\Big\downarrow{\psi_q^*} & & \Big\downarrow{\psi_q^*} \\
\bar{\mathcal{H}}_q & \xrightarrow{\quad\bar{\Omega}\quad} & \bar{\mathcal{H}}_q
\end{array}
$$

where $\Omega \in \mathrm{End}(\mathcal{H}_q)$ and $\bar{\Omega} \in \mathrm{End}(\bar{\mathcal{H}}_q)$. We have

$$\bar{\Omega}(\bar{\xi}) = \Omega\xi \circ \psi_q$$

where $\bar{\xi} = \xi \circ \psi_q$, and the mapping $J$ is defined by $J(\Omega) = \bar{\Omega}$. We thus have a "local (or algebroid) version" of the representation $\pi_q$ of the algebra $\mathcal{A}$, $\bar{\pi}_q : \mathcal{A} \to \mathcal{B}(\bar{\mathcal{H}}_q)$ which is given by

$$\bar{\pi}_q = J \circ \pi_q.$$

By using this representation we can write the "local version" of our dynamical equation (9)
$$i\hbar\bar{\pi}_q(v(a)) = [F_v, \bar{\pi}_q(a)].$$

The Hilbert space $\mathcal{H}_q = L^2(T_e\Gamma_q)$ is now defined on the space $\mathbf{R}^n$. To complete the transition to the ordinary quantum mechanics let us consider the equivalence relation

$$(p_1, g_1) \sim (p_2, g_2) \Leftrightarrow \exists_{g \in G} p_2 = p_1 g.$$

The function $\varphi \in \mathcal{A}$ will be called *G-invariant* if

$$(p_1, g_1) \sim (p_2, g_2) \Rightarrow \varphi(p_1, g_1) = \varphi(p_2, g_2).$$

The set of all such functions forms a subalgebra $\mathcal{A}_{inv}$ of $\mathcal{A}$. Accordingly, a derivation $v \in V_1$ will be called *G-invariant* if there exist $\varphi \in \mathcal{A}$ such that

$v(\varphi) \in \mathcal{A}_{inv}$. The set of all $G$-invariant derivations of $\mathcal{A}$ will be denoted by $V_{inv}$.

Let $\frac{\partial}{\partial x_i}$ be a basis in $\mathbf{R}^n$ $(= T_e G)$. Then our dynamical equation assumes the form

$$i\hbar \widehat{\frac{\partial a}{\partial x^i}} = [F_{\widehat{\frac{\partial}{\partial x_i}}}, \bar{\pi}(a)]$$

where $\widehat{\frac{\partial a}{\partial x^i}} = \bar{\pi}_q(\frac{\partial}{\partial x^i}(a))$. If we put $F_{\widehat{\frac{\partial}{\partial x^0}}} = H$ where $H$ is bounded and positive then we can identify it with the Hamiltonian of the system, and we finally obtain

$$i\hbar \widehat{\frac{\partial a}{\partial x^0}} = [H, \hat{a}],$$

The remaining components give us

$$i\hbar \widehat{\frac{\partial a}{\partial x^i}} = [F_{\widehat{\frac{\partial}{\partial x^i}}}, \hat{a}],$$

with $i = 1, 2, \ldots, n$, and $\hat{a} = \bar{\pi}(a)$. The last two equations are the well known quantum mechanical equations for the evolution of energy and momentum, respectively.

## 6.3 Transition: Quantum Mechanics – Classical Mechanics

The transition form quantum mechanics to classical mechanics is a standard problem which can be discussed (and is widely discussed) beyond the framework of our model. It turns out, however, that our model places this transition in a transparent conceptual setting and within a natural mathematical structure. The idea is simple: if in our model we go from the groupoid $\Gamma$ to its tangent groupoid $\mathcal{G}_\Gamma$ then our formulation of quantum mechanics goes to the usual classical mechanics. Although this procedure is known and was applied to the standard formulation of quantum mechanics (see, for example, Ref. 15, pp. 78-84), we repeat it briefly for the sake of completeness of this presentation.

Let us define $\Gamma_\epsilon := (\Gamma \times \Gamma) \times \epsilon$ where $\epsilon \in \mathbf{R} \setminus \{0\}$, and $\Gamma_0 := T\Gamma \times \{0\}$ where $T\Gamma$ is the tangent bundle. The *tangent groupoid* is defined to be

$$\mathcal{G}_\Gamma := \bigcup_\epsilon \Gamma_\epsilon \cup \Gamma_0$$

with the usual topologies on $\Gamma_\epsilon$ and $\Gamma_0$, and the condition that if $\epsilon \to 0$ then $\Gamma_\epsilon \to \Gamma_0$ in the following sense: for any sequence $(\gamma_n, \eta_n, \epsilon_n)$ such that $\gamma_n, \eta_n \in \Gamma$, if $\lim_{n\to\infty} \eta_n = \eta = \lim \gamma_n$, $\lim_{n\to\infty} \epsilon_n = 0$, and $\lim_{n\to\infty} \frac{\gamma_n - \eta_n}{\epsilon_n} = X \in T\Gamma$, then

$$(\gamma_n, \eta_m, \epsilon_n) \to (\eta, X, 0) \in T\Gamma \times \{0\}.$$

It can be shown that the $C^*$-algebra of the groupoid $\Gamma_\epsilon$ is the algebra $\mathcal{K}(L^2(\Gamma))$ of compact operators on the separable Hilbert space $L^2(\Gamma)$, and the $C^*$-algebra of the groupoid $\Gamma_0$ is $C_0(T^*\Gamma)$ (Ref. 15, p. 79). The tangent groupoid structure gives us the following deformation of groupoids $\Gamma_\epsilon \xrightarrow{\epsilon \to 0} \Gamma_0$ and, consequently, the following deformation of their $C^*$-algebras

$$\mathcal{K}(L^2(\Gamma)) \xrightarrow{\epsilon \to 0} C_0(T^*\Gamma).$$

Let us notice that we have the inclusion

$$\beta : C_0(T^*\Gamma) \hookrightarrow C_0(T^*M).$$

Indeed, the natural projection $pr_M : \Gamma \to M$ leads to the mapping

$$pr_M^* : T^*M \to T^*\Gamma$$

such that to each function $f \in C_0(T^*\Gamma)$ there corresponds the function $f \circ \pi_M^* \in C_0(T^*M)$.

If we now choose the subalgebra $\tilde{\mathcal{A}}$ of our algebra $\mathcal{A}$ such that $\pi(\tilde{\mathcal{A}}) = \mathcal{K}(L^2(\Gamma))$, where $\pi$ is the representation (5) of $\mathcal{A}$, we obtain the deformation

$$\pi(\tilde{\mathcal{A}}) = \mathcal{K}(L^2(\Gamma)) \xrightarrow{\epsilon \to 0} C_0(T^*\Gamma) \xrightarrow{\beta} C_0(T^*M).$$

If we notice that $\mathcal{K}(L^2(\Gamma))$ is the algebra of observables of the quantum sector of our model, and $C_0(T^*M)$ the algebra of observables of classical mechanics, it is enough to identify the parameter $\epsilon$ with the Planck constant $\hbar$ to obtain the transition from the quantum sector of our model to classical mechanics. Let us also notice that

$$C_0(T^*M) \setminus \beta(C_0(T^*\Gamma)) \neq \emptyset$$

implies that there exist classical observables that do not come from compact quantum operators; for instance, position and momentum in quantum mechanics are non compact operators.

# 7 Noncommutative Fock Space

The model constructed in the previous sections gives us a conceptually transparent unification of general relativity and quantum mechanics. However, its main drawback is that it is lacking the quantum field theoretical aspect. To at least partially improve this situation and to indicate the line of the future development, in the present Section, we construct the Fock space for our model.

As it is well known, Fock space is a phase space for a quantum system of many identical, noninteracting particles which in the following will also be called quanta. Let $\mathcal{H}$ be a space of states of a single quantum. In our model $\mathcal{H}$ is $L^2(\Gamma_p)$ or, if needed, $L^2(\Gamma) = \bigoplus_{p \in E} L^2(\Gamma_p)$, $p \in E$. We assume that $\mathbf{C}$ is the space of vacuum states, i. e., the space of "zero quantum states". Let us define the direct sum of the following spaces (as linear spaces) $F = \mathbf{C} \oplus \mathcal{H} \oplus (\mathcal{H} \otimes \mathcal{H}) \oplus (\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}) \oplus \cdots$ ($\mathbf{C}$ and $\mathcal{H}$ are regarded here as linear spaces). This space would correspond to a superposition of states having different numbers of quanta (no quantum, one quantum, two quanta...), but it is too large; we must correct it by taking into account that quanta are indistinguishable. To this end we define the *symmetrization operator* $\sigma = \mathcal{H} \otimes \cdots \otimes \mathcal{H} \to \mathcal{H} \otimes \cdots \otimes \mathcal{H}$ by

$$\sigma(\xi_1 \otimes \cdots \otimes \xi_n) = \frac{1}{n!} \sum_{s \in \Pi_n} \xi_{s(1)} \otimes \cdots \otimes \xi_{s(n)}$$

where $\Pi_n$ is the set of $n$ elementary permutations; and the *antisymmetrization operator* $\tau = \mathcal{H} \otimes \cdots \otimes \mathcal{H} \to \mathcal{H} \otimes \cdots \otimes \mathcal{H}$ by

$$\tau(\xi_1 \otimes \cdots \otimes \xi_n) = \frac{1}{n!} \sum_{s \in \Pi_n} \text{sign}(\xi_{s(1)} \otimes \cdots \otimes \xi_{s(n)}).$$

Of course, $\sigma \circ \sigma = \sigma$, and $\tau \circ \tau = \tau$. An element $w \in \mathcal{H} \otimes \cdots \otimes \mathcal{H}$ is *symmetric* if $\sigma(w) = w$, and *antisymmetric* if $\tau(w) = w$. Let us denote by $F_\sigma$ the subspace of the vector space $F$ all components of which are symmetric, and by $F_\tau$ the subspace of $F$ all components of which are antisymmetric. $F_\sigma$ is called the *symmetric Fock space,* and it describes bosonic quantum states. $F_\tau$ is called the *antisymmetric Fock space*, and it describes fermionic quantum states.

So far it was simply a repetition of the well known construction of the Fock space for the case when $\mathcal{H} = L^2(\Gamma_p)$ or $\mathcal{H} = L^2(\Gamma)$. The rest is also

straightforward, but it requires a certain care. We must check whether the analogous construction can be carried out for the algebra $\mathcal{A}$ (if necessary regarded as a vector space).

First, let us consider the tensor product $\mathcal{A} \otimes \mathcal{A}$. Addition in this algebra is defined as usual: homogeneous elements are added in the ordinary way whereas nonhomogeneous elements form the internal direct sum. Involution is defined in the natural way: $(a_1 \otimes a_2)^* = a_2^* \otimes a_1^*$. If $\mathcal{A}$ is regarded as a vector space, one can define the symmetrization and antisymmetrization operations as above.

For any derivation $v \in \mathrm{Der}\mathcal{A}$ we define the derivation of the algebra $\mathcal{A}^p$, $\bar{v} : \mathcal{A}^p \to \mathcal{A}^p$ by

$$\bar{v}(a_1 \otimes \cdots \otimes a_p) = \sum_{i=1}^{p} a_1 \otimes \cdots \otimes a_{i-1} \otimes v(a_1) \otimes a_{i+1} \otimes \cdots \otimes a_p.$$

Now we can construct the *Fock algebra* in the following way

$$F(\mathcal{A}) = \mathbf{C} \oplus \mathcal{A} \oplus \mathcal{A}^2 \oplus \cdots \oplus \mathcal{A}^p \oplus \cdots$$

In the following we shall consider the "extended module" $\bar{V}$ of derivations of the Fock algebra. We thus have the differential algebra $(F(\mathcal{A}), \bar{V})$ corresponding to the Fock space; we shall call it the *Fock differential algebra*. Basing on this differential algebra we shall construct the geometry of the Fock space.

For any tensor $T : V \times \cdots \times V \to V$ we define its extension $\bar{T} : \bar{V} \times \cdots \times \bar{V} \to \bar{V}$ by

$$\bar{T}(v_1, \ldots, v_k) = \overline{T(v_1, \ldots, v_k)}.$$

From the above we immediately have the following lemma.

**Lemma 7.1** $\bar{T} = 0$ *if and only if* $T = 0$. $\square$

Since the Einstein equation of our model has the form $\mathbf{G} = 0$, where $\mathbf{G}$ is the generalized Einstein tensor, the above lemma implies that the same form of the Einstein equation is valid on the Fock space.

It is straightforward that the representation of the algebra $\mathcal{A}^p$ in the Hilbert space $\mathcal{H}^p = \otimes_{i=1}^{p} \mathcal{H}_i$, $\pi_q : \mathcal{A}^p \to \mathrm{End}\mathcal{H}^p$ should be defined by

$$\pi_q(a_1 \otimes \cdots \otimes a_p) := \pi_q(a_1) \otimes \cdots \otimes \pi_q(a_p)$$

where $\pi_q(a_i)$, $i = 1, \ldots, p$, is the representation of the algebra $\mathcal{A}$ in the Hilbert space $\mathcal{H}$ given by (3). For instance, on a simple element we have

$$\pi_q(a_1 \otimes \cdots \otimes a_p)(\xi_1 \otimes \cdots \otimes \xi_p) = \pi_q(a_1)(\xi_1) \otimes \cdots \otimes \pi_q(a_p)(\xi_p) =$$

$$= (a_1 \otimes \cdots \otimes a_p) * (\xi_1 \otimes \cdots \otimes \xi_p).$$

Now, it is easy to write dynamical equation (9); for instance, in gradation 2 we have

$$i\hbar\pi_q(\bar{v}(a_1 \otimes a_2))(\xi_1 \otimes \xi_2) = [F_v \otimes F_v, \pi_q(a_1 \otimes a_2)](\xi_1 \otimes \xi_2)$$

with $\bar{v} \in \ker\bar{\mathbf{G}}$ and all other symbols self-evident.

An element $\xi_1 \otimes \cdots \otimes \xi_p$ is said to be *G-invariant* if all its components are $G$-invariant (i. e., if they are constant on the equivalence classes of the action of the group $G$). Let $a = (a_1 \otimes \cdots \otimes a_p) \in \mathcal{A}_G \otimes \cdots \otimes \mathcal{A}_G$, and let $\tilde{\xi}_1 \otimes \cdots \otimes \tilde{\xi}_p$ be $G$ invariant. In such a case $\tilde{\xi}_1 \otimes \cdots \otimes \tilde{\xi}_p$ can be regarded as an element of the Fock space $F(L^2(\Gamma_q))$, and the equation

$$i\hbar\pi_q(\bar{v}(a_1 \otimes \cdots \otimes a_p))(\tilde{\xi}_1 \otimes \cdots \otimes \tilde{\xi}_p) =$$

$$[F_v \otimes \cdots \otimes F_v, \pi_q(a_1 \otimes \cdots \otimes a_p)](\tilde{\xi}_1 \otimes \cdots \otimes \tilde{\xi}_p)$$

space.

The standard operators on the above noncommutative Fock space can be defined in close analogy to the usual case. First, we define the *number operator* $N : F(\mathcal{H}) \to F(\mathcal{H})$, by

$$\xi = (\xi_0, \xi_1, \xi_2, \ldots) \mapsto N(\xi) = (0, 1, \xi_1, 2\xi_2, 3\xi_3, \ldots)$$

where $\xi_i \in \bigotimes_{i=1}^{p} \mathcal{H}^p$. It can be easily seen that there exists $(a_0, a_1, a_2, \ldots) \in F(\mathcal{A})$, where $a^p \in \mathcal{A}$, such that $\pi_q(a_i) = \xi_i$, and we have the corresponding operator in the Fock algebra $\tilde{N} : F(\mathcal{A}) \to F(\mathcal{A})$ defined by

$$\tilde{N}(a_0, a_1, a_2, \ldots) = (0, 1a, 2a, 3a, \ldots).$$

If $\xi$ is in the symmetric Fock space $F_\sigma$, so is $N(\xi)$; the same refers to the antisymmetric Fock space $F_\tau$. Therefore, in fact we have two linear operators: $N_\sigma$ in $F_\sigma$, and $N_\tau$ in $F_\tau$. For instance, for $\xi \in F_\sigma$, one has $N(\xi) = 3$ if and only if $\xi = (0, 0, 0, \xi_3)$ which means that the system is in the state corresponding to 3 bosons.

Let us fix $\eta \in \mathcal{H} = L^2(G_q)$, and let us define the *creation operator* $C_\eta : F(\mathcal{H}) \to F(\mathcal{H})$ by

$$C_\eta(\xi) = (0, \sqrt{1}\xi_0\eta, \sqrt{2}\eta \otimes \xi_1, \sqrt{3}\eta \otimes \xi_2, \ldots),$$

and correspondingly for the Fock algebra $\tilde{C}_a : F(\mathcal{A}) \to F(\mathcal{A})$ by

$$\tilde{C}_b(a_0, a_1, a_2, a_3, \ldots) = (0, \sqrt{1}a_0 b, \sqrt{2}b \otimes a_1, \sqrt{3}b \otimes a_2, \ldots)$$

where $a$ is a fixed element of $\mathcal{A}$. It can be checked that

$$N \circ C_\eta = C_\eta \circ (N + I_F),$$

where $I_F$ is the identity on $F$; and analogously

$$\tilde{N} \circ \tilde{C}_b = \tilde{C}_b \circ (\tilde{N} + I_{F(\mathcal{A})}).$$

The above formula says that the action of the creation operator increases the number of quanta by one.

Let $\phi$ be an element of the dual $\mathcal{H}^*$ to the Hilbert space $\mathcal{H}$. The *annihilation operator* $A_\phi : F(\mathcal{H}) \to F(\mathcal{A})$ is defined by

$$A_\phi(0, 0, \ldots, \zeta \otimes \zeta_1 \otimes \cdots \otimes \zeta_{n-1}, 0, 0) = (0, 0, \ldots \sqrt{n}\phi(\zeta) \otimes \cdots \otimes \zeta_{n-1}, 0, 0, \ldots),$$

and then one extends (by linearity) the operator $A_\phi$ to the whole space $F(\mathcal{H})$. Analogously one defines the corresponding annihilation operator for the Fock algebra $\tilde{A}_\varphi : F(\mathcal{A}) \to F(\mathcal{A})$ by

$$\tilde{A}_\varphi(0, 0, \ldots, b \otimes a_1 \otimes \cdots \otimes a_{n-1}, 0, 0) = (0, 0, \ldots, \sqrt{n}\phi(b)a_1 \otimes \cdots \otimes a_{n-1}, 0, 0 \ldots)$$

for a fixed element $\varphi \in \mathcal{A}^*$. One also has

$$(\tilde{N} + I_{F(\mathcal{A})}) \circ \tilde{A}_\varphi = \tilde{A}_\varphi \circ \tilde{N}$$

(and similarly for the space $F(\mathcal{H})$) which says that the action of the annihilation operator diminishes the number of quanta by one.

It can be easily checked that the following (anti)commutation relations (analogous to those known from the standard field theory) are valid:

For the bosonic sector

$$[\tilde{C}_b, \tilde{C}_{b'}] = 0, \quad [\tilde{A}_\varphi, \tilde{A}_{\varphi'}] = 0, \quad [\tilde{A}_\varphi, \tilde{C}_b] = [\varphi(b)]I_{F_\sigma(\mathcal{A})},$$

and for the fermionic sector

$$[\tilde{C}_b, \tilde{C}_{b'}]_+ = 0, \quad [\tilde{A}_\varphi, \tilde{A}_{\varphi'}]_+ = 0, \quad [\tilde{A}_\varphi, \tilde{C}_b] = [\varphi(b)]I_{F_\tau(\mathcal{A})}.$$

# 8 Concluding Remark

The model developed in the present paper creates a transparent conceptual framework for a unification of general relativity and quantum mechanics. It "predicts" nonlocal phenomena present in both these theories, such as the horizon paradox and the EPR experiment; it explains the state vector reduction, and naturally unifies probability and dynamics (both generalized probability and dynamics are incorporated into von Neumann algebras, see Ref. 2). It also has the correct correspondence with general relativity, quantum mechanics, and classical mechanics. And the construction of the Fock space for this model suggests that it can be enlarged to include the field theoretical aspect. Two elements, on which the model is based, are responsible for these results. First of them is obviously the idea of noncommutative geometry. It has made the model consequently nonlocal and enabled the generalization of many geometrical concepts. The second element is the groupoid structure. Owing to it the standard physical concept of symmetry has been replaced by a generalized symmetry. The generalization is at least twofold: first, groupoids can be thought of as "groups with many identities"; second, an algebraic structure underlying geometry based on a group is a set with a distinguished base point, whereas the analogous structure underlying a groupoid geometry is a directed graph [26]. This opens new possibilities for a physical theory. Some of them were explored in the present paper.

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